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A REFINED THEORY OF LAMINATED SHELLS WITH HIGHER-ORDER TRANSVERSE SHEAR DEFORMATION

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Abstract—Based on refined analysis, a theory of laminated composite shells with higher-order transverse shear deformation is presented. The continuity conditions of displacements and transverse shear stresses at layer interfaces and the conditions of zero transverse shear stresses on the surfaces of the shells are introduced to improve and simplify the displacement field. The number of the displacement unknowns and the order of the equilibrium equations are the same as in the first-order shear deformation theory, but the present theory can predict continuous parabolic transverse shear stresses. The closed-form solutions of simply-supported cross-ply shells are obtained and compared with the elasticity solutions and other theories' solutions. For both shallow and deep shells, the present solutions of displacements and in-plane stresses are very close to the elasticity solutions. Copyright © 1997 Elsevier Science Ltd.

1. INTRODUCTION

The early researches of laminated composite shells were based on the classical shell theory (CPT) and the first order shear deformation theory (FSDT) which are suited to homogeneous shells. Because of the low ratios of the transverse shear modulus to the in-plane modulus in composite laminates, the effects of transverse shear deformation on the responses of laminated composite shells must be considered fully. In the past three decades, various theories for laminated shells have been presented. These theories can be divided to two kinds, i.e., the piecewise approximation theories and the global approximation theories. Theories of the first kind, because the order of the governing equations varies with the number of layers, their application is limited.

In the theories of the second kind, the displacements are expanded as functions of the thickness coordinate. The order of governing equations is independent of the number of layers. Some first-order and high-order shear deformation shell theories were proposed by Dong and Tso (1972), Reddy and Liu (1985), Librescu *et al.* (1989), Dennis and Palazotto (1991), etc. But the continuity conditions of transverse shear stresses at layer interfaces in these theories are not fulfilled and the continuous transverse shear stresses can be obtained only by integrating the three-dimensional equilibrium equations. Although some authors thought that their global theories did not need shear correction factors, shear correction factors are often necessary (Huang, 1994; Noor and Peter, 1989; etc.). To overcome this drawback, Di Sciuva (1987) propsed a simplified discrete-layer theory with five unknowns which ensure the continuity of transverse shear stresses at layer interfaces. But in this theory, the transverse shear stresses are uniform across the thickness of the shell, therefore the compatibility conditions on the external bounding surfaces are not fulfilled. Soldatos and Timarci (1993) gave unified formulation of 'laminated composite, shear deformation, five-degrees-of-freedom cylindrical shell theory'. In their theory, 'shear deformation shape function' may be chosen and open possibilities are left for posterior specification of particular shear deformable shell theories. Recently Jing and Tzeng (1993) and He (1994) have presented the refined shell theories (He, five independent displacement unknowns; Jing and Tzeng, seven) in which the continuity conditions of interlaminar transverse shear stresses and the compatibility conditions on the external bounding surfaces are fulfilled, but the form of the displacement fields is quite complicated. Moreover, it should be noted that many of the theories were presented for laminated shallow shells. The Donnell simplification for shallow shells (omitting the terms u/R_1 and v/R_2 in the transverse shear strains) is adopted in these theories. Consequently, considerable errors are yielded for deep shells ($R/l < 3$).

Shu (1994a) presented a simple higher-order theory for laminated composite plates. The present study is to present a theory of laminated shells (a global approximation theory). The new contribution in the research area of the paper is based on the following points: (1) transverse shear stresses are considered continuous at layer interfaces. By ensuring the continuity of interlaminar transverse shear stresses and the zero transverse shear strains on the surface of shells, the number of the displacement parameters is reduced to five which is the same as in FSDT. The influence of the materials and plyup patterns of shells on the displacement field is considered. (2) The theory is presented for general shells. (3) Some interesting results are presented for cross-ply laminated shells of cylindrical and spherical shape. The present theory can predict very accurate responses for both shallow and deep shells. Moreover, it is relatively simple for solution.

2. REFINED DISPLACEMENT FIELD

Consider a laminated shell composed of N orthotropic layers with uniform thickness. Let (ξ_1, ξ_2, ξ_3) denote the orthogonal curvilinear coordinates such that ξ_1- and ξ_2-curves are lines of curvature on midsurface $\xi_3 = 0$, and ξ_3-curves (also referred to as z) are straight lines perpendicular to the midsurface. $z = h/2$ and $z = -h/2$ are the top surface and bottom surface of the shell. z_j ($j = 0, 1 \ldots N$) is the z coordinate of each layer interface. The reference surface Ω coincides with the midsurface. The radii of curvatures along ξ_1 and ξ_2 curves are R_1 and R_2, respectively. The Lamé parameters of the midsurface are denoted as A_1 and A_2.

The strain-displacement relations of shells are defined

$$
\begin{aligned}
\varepsilon_1 &= u_{,1}/A_1 + vA_{1,2}/A_1A_2 + w/R_1 \\
\varepsilon_2 &= v_{,2}/A_2 + uA_{2,1}/A_1A_2 + w/R_2 \\
\varepsilon_3 &= w_{,3} \\
\varepsilon_4 &= w_{,2}/A_2 + v_{,3} - \underline{v_0/R_2} \\
\varepsilon_5 &= w_{,1}/A_1 + u_{,3} - \underline{u_0/R_1} \\
\varepsilon_6 &= (A_2/A_1)(v/A_2)_{,1} + (A_1/A_2)(u/A_1)_{,2}
\end{aligned} \tag{1}
$$

where a comma denotes differentiation with respect to the subscript. u_0 and v_0 are the corresponding midsurface displacements in the ξ_1 and ξ_2 directions. The Love first-order geometric approximation (neglecting ξ_3/R_1 and ξ_3/R_2) is invoked. The Donnell simplification of shallow shells can be accomplished by omitting the underlined terms (u_0/R_1 and v_0/R_2) in ε_4 and ε_5. However, the underlined terms are necessary for deep shells.

The stress-strain relations for the ith layer are

$$
\begin{aligned}
\{\sigma\} &= [\sigma_1\sigma_2\sigma_6]^{\mathrm{T}} = [Q_{1i}]\{\varepsilon\} \\
\{\tau\} &= [\sigma_5\sigma_4]^{\mathrm{T}} = [Q_{2i}]\{\gamma\}
\end{aligned} \tag{2}
$$

where $\{\sigma\}$ and $\{\tau\}$ are in-plane stresses and transverse shear stresses, respectively. $\{\varepsilon\}$ and $\{\gamma\}$ are in-plane strains and transverse shear strains, respectively. $[Q_{1i}]$ and $[Q_{2i}]$ are the plane-stress-reduced elastic constant matrix and transverse shear elastic constant matrix for the ith layer, respectively.

As is suggested by Reddy and Liu (1985) and Librescu *et al.* (1989), the in-plane displacements $\{u\}$ are expressed as cubic functions of the z coordinate and the deflection w is independent of the z coordinate. However, in order to ensure the continuity of transverse shear stresses at layer interfaces, $\{u_0\}$ and $\{\theta\}$ (global unknowns) suggested by the above authors are replaced by $\{u_i\}$ and $\{\theta_i\}$ (unknowns of the ith layer) as follows:

$$\{u\} = \{u_i\}+z\{\theta_i\}+z^2\{\varphi\}+z^3\{\psi\} \quad i = 1,2\ldots N \quad w(\xi_1,\xi_2,z) = w(\xi_1,\xi_2). \tag{3}$$

Relative transverse shear strains in the ith layer are

$$\{\gamma_i\} = \{\theta_i\}+2z\{\varphi\}+3z^2\{\psi\}+\{w'\}-\{k_0\}. \tag{4}$$

In eqns (3) and (4),

$$\begin{aligned}
\{u\} &= [u(\xi_1,\xi_2,z)\, v(\xi_1,\xi_2,z)]^{\mathrm{T}} \\
\{u_i\} &= [u_i(\xi_1,\xi_2)\, v_i(\xi_1,\xi_2)]^{\mathrm{T}} \\
\{\theta_i\} &= [\theta_{1i}(\xi_1,\xi_2)\, \theta_{2i}(\xi_1,\xi_2)]^{\mathrm{T}} \\
\{\varphi\} &= [\varphi_1(\xi_1,\xi_2)\, \varphi_2(\xi_1,\xi_2)]^{\mathrm{T}} \\
\{\psi\} &= [\psi_1(\xi_1,\xi_2)\, \psi_2(\xi_1,\xi_2)]^{\mathrm{T}} \\
\{w'\} &= [w(\xi_1,\xi_2)_{,1}/A_1\, w(\xi_1,\xi_2)_{,2}/A_2]^{\mathrm{T}} \\
\{k_0\} &= [u_0/R_1\, v_0/R_2].
\end{aligned} \tag{5}$$

It is seen that the number of displacement functions $\{u_i\}$ and $\{\theta_i\}$ varies with the number of layers. By imposing the following three conditions, the functions $\{u_i\}$, $\{\theta_i\}$, $\{\varphi\}$ and $\{\psi\}$ will be determined in terms of global displacement unknown and the displacement unknowns will be reduced to five.

1. The transverse shear stresses $\{\tau\}$ vanish on the top and bottom surfaces of the shell. So $\{\varphi\}$ and $\{\psi\}$ are determined

$$\begin{aligned}
\{\varphi\} &= (\{\theta_1\}-\{\theta_N\})/2h \\
\{\psi\} &= -2(\{\theta_1\}+\{\theta_N\}+2\{w'\}-2\{k_0\})/3h^2.
\end{aligned} \tag{6}$$

2. The transverse shear stresses $\{\tau\}$ at each layer interface must be continued. At the interface ($z = z_{i-1}$) between $i-1$ layer and i layer

$$[Q_{2i-1}]\{\gamma_{i-1}(z_{i-1})\} = [Q_{2i}]\{\gamma_i(z_{i-1})\} \tag{7}$$

3. The in-plane displacements $\{u\}$ at each layer interface must be continued.

$$\{u_{i-1}\}+z_{i-1}\{\theta_{i-1}\} = \{u_i\}+z_{i-1}\{\theta_i\}. \tag{8}$$

Satisfying the above conditions, the displacement field (2) becomes

$$\{u\} = \{u_0\}+[R(z)]_i\{\theta\}+([R(z)]_i-z[\mathbf{I}])(\{w'\}-\{k_0\}) \tag{9}$$

where $\{u_0\}$ denotes the corresponding midsurface displacements. $\{\theta\} = \{\theta_1\}$. [**I**] is the unit matrix. When let $[R(z)]_i = [0]$, z[**I**] and $(z\text{-}4z^3/3h^2)$[**I**], eqn (9) will have an identical form to those shown in the classical theory, the first-order shear deformation theory and higher-order theory (Reddy and Liu, 1985; Huang, 1994), respectively. For angle-ply shells, $R_{12}(z)$ and $R_{21}(z)$ in $[R(z)]$ do not equal zero. There exists a coupling between the displacement u in the ξ_1 direction and the displacement unknowns v_0 and θ_2 in the ξ_2 direction, and between the displacement v in ξ_2 direction and the displacement unknowns u_0 and θ_1 in the ξ_1 direction. For cross-ply shells, $R_{12}(z)$ and $R_{21}(z)$ equal zero and the in-plane displacements have a simpler form

$$u = u_0 + R_{11}(z)\theta_1 + (R_{11}(z) - z)(w_{,1}/A_1 - u_0/R_1)$$
$$v = v_0 + R_{22}(z)\theta_2 + (R_{22}(z) - z)(w_{,2}/A_2 - v_0/R_2). \quad (10)$$

$[R(z)]_i$ is a coefficient matrix related to materials and plyup pattern:

$$[R(z)]_i = [R_1]_i + z[R_2]_i + z^2[R_3] + z^3[R_4] \quad (11)$$

and

$$[R_3] = 4[h[C_1]_N + 4[C_2]_N]^{-1}[C_2]_N/h$$
$$[R_4] = -4[h[C_1]_N + 4[C_2]_N]^{-1}[C_1]_N/3h$$
$$[R_2]_i = 2([Q_{2i}]^{-1}[C_1]_i - z_i[\mathrm{I}])[R_3] + 3(2[Q_{2i}]^{-1}[C_2]_i - z_i^2[\mathrm{I}])[R_4]$$
$$[R_1]_i = \sum_{j=2}^{i} z_{j-1}([R_2]_{j-1} - [R_2]_j) - \sum_{j=2}^{k} z_{j-1}([R_2]_{j-1} - [R_2]_j)$$
$$[C_1]_i = \sum_{j=1}^{i} [Q_{2j}](z_j - z_{j-1})$$
$$[C_2]_i = \sum_{j=1}^{i} [Q_{2j}](z_j^2 - z_{j-1}^2)/2 \quad (12)$$

where k is the ordinal number of the layer in which the midsurface is located.

It can be seen that the displacement field of the present theory contains five unknowns ($u_0, v_0, w, \theta_1, \theta_2$) which are similar to those in the first-order shear deformation theory and some higher-order theories. However, the in-plane displacements in the present theory are cubic functions of thickness coordinate and fulfill the conditions which are not totally fulfilled in other theories.

3. EQUILIBRIUM EQUATIONS

To simplify analysis, let $dx_1 = A_1 d\xi_1$; $dx_2 = A_2 d\xi_2$. For cylindrical shells, the substitution is exact. For other shells (e.g., spherical shells), it is approximate, but very accurate when $R/a > 1$ (see numerical example). Therefore the strain components associated with eqn (9) are

$$\{\varepsilon\} = \{\varepsilon_0\} + [T_1(z)]_i\{K\} + [T_2(z)]_i\{\theta'\}$$
$$\{\gamma\} = [R(z)]_{,z}(\{\theta\} + \{w'\} - \{k_0\}) \quad (13)$$

where

$$\{\varepsilon_0\} = [u_{0,1} + w/R_1 \;\; v_{0,2} + w/R_2 \;\; u_{0,2} + v_{0,1}]^{\mathrm{T}}$$
$$\{K\} = [-w_{,11} + u_{0,1}/R_1 \;\; -w_{,12} + u_{0,2}/R_1 \;\; -w_{,12} + v_{0,1}/R_2 \;\; -w_{,22} + v_{0,2}/R_2]^{\mathrm{T}}$$
$$\{\theta'\} = [\theta_{1,1}\theta_{1,2}\theta_{2,1}\theta_{2,2}]^{\mathrm{T}}$$
$$[T_1(z)]_i = \begin{bmatrix} z - R_{11i} & 0 & -R_{12i} & 0 \\ 0 & -R_{21i} & 0 & z - R_{22i} \\ -R_{21i} & z - R_{11i} & z - R_{22i} & -R_{12i} \end{bmatrix}$$

$$[T_2(z)]_i = \begin{bmatrix} R_{11i} & 0 & R_{12i} & 0 \\ 0 & R_{21i} & 0 & R_{22i} \\ R_{21i} & R_{11i} & R_{22i} & R_{12i} \end{bmatrix}. \tag{14}$$

Here R_{11i}, R_{12i}, R_{21i} and R_{22i} are the elements in matrix $[R(z)]_i$.

The principle of virtual work for the present case is

$$\int_h \left(\int_\Omega [\{\sigma\}^T \delta\{\varepsilon\} + \{\tau\}^T \delta\{\gamma\}] \, dx_1 \, dx_2 \right) dz - \int_\Omega (q^+ + q^-) \, \delta w \, dx_1 \, dx_2$$

$$= \int_\Omega [\{N\}^T \delta\{\varepsilon_0\} + \{M\}^T \delta\{K\} + \{S\}^T \delta\{\theta'\} + \{V\}^T (\delta\{\theta\} + \delta\{w'\} - \delta\{k_0\})$$

$$- (q^+ + q^-) \delta w] \, dx_1 \, dx_2 = 0. \tag{15}$$

The equilibrium equations and boundary conditions are obtained from the principle of virtual work. The equilibrium equations are

$$\begin{aligned} N_{1,1} + N_{12,2} + \underline{M_{1,1}/R_1 + M_{12,2}/R_1 + V_1/R_1} &= 0 \\ N_{12,1} + N_{2,2} + \underline{M_{21,1}/R_2 + M_{2,2}/R_2 + V_2/R_2} &= 0 \\ M_{1,11} + (M_{12} + M_{21})_{,12} + M_{2,22} + V_{1,1} + V_{2,2} - N_1/R_1 - N_2/R_2 + q^+ + q^- &= 0 \\ S_{1,1} + S_{12,2} - V_1 &= 0 \\ S_{21,1} + S_{2,2} - V_2 &= 0. \end{aligned} \tag{16}$$

The equivalent surface tractions q^+ and q^- in eqn (16) are related with prescribed tractions p^+ on the surface $z = h/2$ and traction p^- on the other surface $z = -h/2$ as follows:

$$\begin{aligned} q^+ &= p^+ (1 + 0.5h/R_1)(1 + 0.5h/R_2) \\ q^- &= p^- (1 - 0.5h/R_1)(1 - 0.5h/R_2). \end{aligned} \tag{17}$$

The stress resultants can be expressed in terms of strain components

$$\begin{aligned} \{N\} &= [N_1 N_2 N_{12}]^T = \int_h \{\sigma\} \, dz = [A]\{\varepsilon_0\} + [B]\{K\} + [F]\{\theta'\} \\ \{M\} &= [M_1 M_{12} M_{21} M_2]^T = \int_h [T_1(z)]^T \{\sigma\} \, dz = [B]^T \{\varepsilon_0\} + [D]\{K\} + [E]\{\theta'\} \\ \{S\} &= [S_1 S_{12} S_{21} S_2]^T = \int_h [T_2(z)]^T \{\sigma\} \, dz = [F]^T \{\varepsilon_0\} + [E]^T \{K\} + [G]\{\theta'\} \\ \{V\} &= [V_1 V_2]^T = \int_h [R(z)]_{,z}^T \{\tau\} \, dz = [C](\{\theta\} + \{w'\} - \{k_0\}). \end{aligned} \tag{18}$$

Stiffnesses $[A]$, $[B]$, etc. are defined as

$$([A][B][F]) = \int_h [Q_1](1[T_1][T_2])\,\mathrm{d}z$$

$$([D][E]) = \int_h [T_1]^{\mathrm{T}}[Q_1]([T_1][T_2])\,\mathrm{d}z$$

$$[G] = \int_h [T_2]^{\mathrm{T}}[Q_1][T_2]\,\mathrm{d}z$$

$$[C] = \int_h [R(z)]_{,z}^{\mathrm{T}}[Q_2][R(z)]_{,z}\,\mathrm{d}z. \tag{19}$$

The corresponding boundary conditions along edge x_1 = constant are of the form:

$$\begin{aligned}
u_0 &\quad \text{or} \quad N_1 + \underline{M_1/R_1} \\
v_0 &\quad \text{or} \quad N_{12} + \underline{M_{21}/R_2} \\
w &\quad \text{or} \quad M_{1,1} + M_{12,2} + V_1 \\
w_{,1} &\quad \text{or} \quad M_1 \\
\theta_1 &\quad \text{or} \quad S_1 \\
\theta_2 &\quad \text{or} \quad S_{21}.
\end{aligned} \tag{20}$$

The boundary conditions along edge x_2 = constant are analogous to the above expressions.

If the underlined terms in the equilibrium equations and boundary conditions are omitted, the form of the equilibrium equations and boundary conditions will be similar to the corresponding shallow shell theory.

4. THE CLOSED-FORM SOLUTIONS FOR CROSS-PLY SHELLS

Here we consider the closed-form solutions of simply supported, cross-ply rectangular $(a \times b)$ shells having both radii of principal curvature constants. For cross-ply shells, the following stiffnesses are identically zero

$$\begin{aligned}
&A_{13} = A_{23} = 0 \\
&B_{12} = B_{22} = B_{13} = B_{23} = B_{31} = B_{34} = 0 \\
&F_{12} = F_{22} = F_{13} = F_{23} = F_{31} = F_{34} = 0 \\
&E_{12} = E_{13} = E_{21} = E_{24} = E_{31} = E_{34} = E_{42} = E_{43} = 0 \\
&D_{12} = D_{13} = D_{24} = D_{34} = 0 \\
&G_{12} = G_{13} = G_{24} = G_{34} = 0.
\end{aligned} \tag{21}$$

Moreover, for symmetric laminated shells, $[B] = [0]$ and $[F] = [0]$.

The simply supported boundary conditions are assumed to be of the form

$$\begin{aligned}
u_0(x_1,0) &= u_0(x_1,b) = v_0(0,x_2) = v_0(a,x_2) = 0 \\
w(x_1,0) &= w(x_1,b) = w(0,x_2) = w(a,x_2) = 0 \\
\theta_1(x_1,0) &= \theta_1(x_1,b) = \theta_2(0,x_2) = \theta_2(a,x_2) = 0 \\
N_2(x_1,0) &= N_2(x_1,b) = N_1(0,x_2) = N_1(a,x_2) = 0 \\
M_2(x_1,0) &= M_2(x_1,b) = M_1(0,x_2) = M_1(a,x_2) = 0 \\
S_2(x_1,0) &= S_2(x_1,b) = S_1(0,x_2) = S_1(a,x_2) = 0.
\end{aligned} \tag{22}$$

We assume the following Navier solution form that satisfies the simply supported boundary conditions

$$u_0 = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} u_{mn} \cos \alpha x_1 \sin \beta x_2$$

$$v_0 = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} v_{mn} \sin \alpha x_1 \cos \beta x_2$$

$$w = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} w_{mn} \sin \alpha x_1 \sin \beta x_2$$

$$\theta_1 = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \theta_{1mn} \cos \alpha x_1 \sin \beta x_2$$

$$\theta_2 = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \theta_{2mn} \sin \alpha x_1 \cos \beta x_2 \tag{23}$$

where $\alpha = m\pi/a$ and $\beta = n\pi/b$.

The transverse loads can be expanded in the double-Fourier series

$$(q^+ q^-) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (q_{mn}^+ q_{mn}^-) \sin \alpha x_1 \sin \beta x_2. \tag{24}$$

Substituting eqns (23) and (24) into equilibrium eqns (16), collecting the coefficients, we obtain

$$[K_{ij}]\{\chi\} = \{Q\} \quad i,j = 1,\ldots,5 \tag{25}$$

where

$$\{\chi\} = [u_{mn} v_{mn} w_{mn} \theta_{1mn} \theta_{2mn}]^{\mathrm{T}}$$

$$\{Q\} = [0 \;\; 0 \;\; q_{mn}^+ + q_{mn}^- \;\; 0 \;\; 0]^{\mathrm{T}}. \tag{26}$$

$[K_{ij}]$ is the coefficient matrix and is listed in the Appendix.

5. NUMERICAL EXAMPLES AND DISCUSSIONS

In order to assess the accuracy of the present theory and determine its application range, we present numerical results for simply-supported cross-ply rectangular shells. The solutions of the present theory are compared with the elasticity solutions and the solutions of other higher-order shell theories. Cylindrical and spherical shells are examined:

(a) cylindrical shells: $b/a = 3$. Deep shell, $R_1/a = 1$; shallow shell, $R_1/a = 4$;
(b) spherical shells: $b/a = 1$. Deep shell, $R/a = 1, 2$; shallow shell, $R/a = 5$.

The lamination scheme are of symmetric $[0/90\ldots]_s$ ($N = 3, 5$) and antisymmetric $[0/90/0/90\ldots]$ ($N = 4$) type.

In all problems, the following lamina material properties and the transverse loads are used:

$$E_{11}/E_{22} = 25 \quad G_{12}/E_{22} = G_{13}/E_{22} = 0.5 \quad G_{23}/E_{22} = 0.2$$

$$\nu_{12} = \nu_{13} = \nu_{23} = 0.25 \quad p^- = 0. \quad p^+ = p \sin(\pi x_1/a) \sin(\pi x_2/b).$$

In all tables, the nondimensionalized deflections and stresses are used:

Table 1. Nondimensional deflections and stresses of laminated cylindrical deep shells ($R_1/a = 1$)

N	a/h	Theory	$\bar{w}$	$\bar{\sigma}_{1-}$	$\bar{\sigma}_{1+}$	$\bar{\sigma}_2 \times 10$	$\bar{\sigma}_6 \times 10$	$\bar{\sigma}_5$	$\bar{\sigma}_4 \times 10$
3	5	Elast.	2.716	−1.293	—	2.411	0.4371	0.4447	0.3442
		$HSDT_1$	2.699	−1.241	1.184	2.379	0.4335	0.4767*	0.3026
		$HSDT_2$	2.195	−1.093	1.104	1.918	0.3588	0.4317*	0.2512
		$HSDT_3$	2.525	−1.094	—	2.228	0.4046	0.4794#	0.3169
	10	Elast.	1.153	−0.8534	—	1.602	0.2725	0.4697	0.1848
		$HSDT_1$	1.145	−0.8279	0.8063	1.589	0.2734	0.4873	0.1702
		$HSDT_2$	0.934	−0.7371	0.7464	1.290	0.2274	0.4422	0.1441
		$HSDT_3$	1.077	−0.7876	—	1.498	0.2581	0.4821	0.1764
4	5	Elast.	3.707	−1.668	—	4.672	0.6597	0.5392	0.8547
		$HSDT_1$	3.775	−1.614	0.1171	4.792	0.6465	0.5640	0.9017
		$HSDT_2$	3.101	−1.459	0.1094	3.922	0.5414	0.5091	0.7503
		$HSDT_3$	3.109	−1.500	—	4.091	0.5475	0.5946	0.7233
	10	Elast.	1.851	−1.222	—	3.314	0.4883	0.5597	0.5567
		$HSDT_1$	1.844	−1.198	0.0820	3.302	0.4828	0.5440	0.6181
		$HSDT_2$	1.539	−1.094	0.0764	2.753	0.4104	0.4961	0.5244
		$HSDT_3$	1.685	−1.176	—	3.049	0.4478	0.5781	0.5028
5	5	Elast.	2.818	−1.301	—	3.132	0.4561	0.4831	0.5104
		$HSDT_1$	2.824	−1.233	1.173	3.134	0.4477	0.4908	0.4767
		$HSDT_2$	2.302	−1.086	1.099	2.540	0.3709	0.4454	0.3937
		$HSDT_3$	2.458	−1.154	—	2.784	0.3983	0.4564	0.4498
	10	Elast.	1.242	−0.9436	—	2.044	0.3030	0.4544	0.2904
		$HSDT_1$	1.235	−0.9169	0.8952	2.030	0.3023	0.4738	0.2752
		$HSDT_2$	1.009	−0.8185	0.8302	1.656	0.2520	0.4309	0.2308
		$HSDT_3$	1.144	−0.9000	—	1.895	0.2839	0.4491	0.2679

$\bar{\sigma}_{1-}$ and $\bar{\sigma}_{1+}$ are calculated at the bottoms and tops of the shells, respectively.
*—σ_4, σ_5 are obtained from constitutive equations.
#—σ_4, σ_5 are obtained from equilibrium equations.

$$\bar{w} = 100E_{22}w(a/2, b/2)/phS^4 \quad \bar{\sigma}_1 = \sigma_1(a/2, b/2, \pm h/2)/pS^2$$

$$\bar{\sigma}_2 = \sigma_2(a/2, b/2, z)/pS^2 \quad (N = 3, z = h/6; N = 4, z = h/2; N = 5, z = 3h/10)$$

$$\bar{\sigma}_6 = \sigma_6(0, 0, -h/2)/pS^2 \quad \bar{\sigma}_4 = \sigma_4(a/2, 0, 0)/pS \quad \bar{\sigma}_5 = \sigma_5(0, b/2, 0)/pS \quad S = a/h.$$

In all the tables, Elast. is the three-dimensional elasticity theory; $HSDT_1$ is the present shell theory; $HSDT_2$ is Shu's shallow shell theory (1996); $HSDT_3$ is the higher-order shell theory (Huang, 1994); $HSDT_4$ is Reddy's shallow shell theory. The results of Elast. and $HSDT_3$, $HSDT_4$ are extracted from Huang (1994).

It is revealed in the numerical results in the tables:

(a) For deep or shallow shells, the present shell theory ($HSDT_1$) gives very accurate solutions of deflections and in-plane stresses. But the higher-order shallow shell theory ($HSDT_4$) and the higher-order deep shell theory ($HSDT_3$) yield much higher errors. This is because $HSDT_3$ and $HSDT_4$ do not fulfill the continuity conditions of interlayer transverse shear stresses. Therefore it is acceptable that an excellent theory of laminated plates and shells must fulfill these conditions.

(b) For deep shells, Shu's shallow shell theory ($HSDT_2$) yields high errors. However, for shallow shells, it gives very accurate solutions of deflection and in-plane stresses. It is obvious that the present shell theory and Shu's shallow shell theory yield almost agreeable results for shallow shells. Therefore we can determine the application range of the two theories. Table 4 gives comparison between $HSDT_1$ and $HSDT_2$. It is shown that when $R/a > 3$ (shallow shells), the shallow shell theory is suitable; when $R/a \leqslant 3$ (deep shells), the present shell theory is necessary.

(c) Continuous σ_4 and σ_5 of the present theory are calculated directly from the constitutive equations. But continuous σ_4 and σ_5 of some global theories (e.g., $HSDT_3$, $HSDT_4$)

Table 2. Nondimensional deflections and stresses of laminated cylindrical shallow shells ($R_1/a = 4$)

N	a/h	Theory	$\bar{w}$	$\bar{\sigma}_{1-}$	$\bar{\sigma}_{1+}$	$\bar{\sigma}_2 \times 10$	$\bar{\sigma}_6 \times 10$	$\bar{\sigma}_5$	$\bar{\sigma}_4 \times 10$
3	5	Elast.	2.118	−1.022	—	1.116	0.2588	0.3867	0.2729
		$HSDT_1$	2.108	−1.054	1.042	1.115	0.2565	0.4119	0.2409
		$HSDT_2$	2.081	−1.040	1.043	1.097	0.2535	0.4094	0.2382
		$HSDT_4$	1.944	−0.913	0.915	1.028	0.2356	0.4118#	0.2479
	10	Elast.	0.9396	−0.7463	—	0.6468	0.1510	0.4271	0.1555
		$HSDT_1$	0.9409	−0.7452	0.7400	0.6486	0.1506	0.4427	0.1449
		$HSDT_2$	0.9292	−0.7369	0.7392	0.6388	0.1489	0.4399	0.1434
		$HSDT_4$	0.8712	−0.6985	0.7007	0.6037	0.1405	0.4344	0.1462
4	5	Elast.	3.042	−1.388	—	3.117	0.4006	0.4924	0.7049
		$HSDT_1$	3.018	−1.422	0.1040	3.067	0.3903	0.4948	0.6790
		$HSDT_2$	2.981	−1.409	0.1040	3.026	0.3861	0.4902	0.6713
		$HSDT_4$	2.494	−1.301	0.0838	2.640	0.3342	0.5443	0.5995
	10	Elast.	1.609	−1.137	—	2.045	0.2822	0.5379	0.4869
		$HSDT_1$	1.594	−1.138	0.0773	2.017	0.2780	0.5140	0.4784
		$HSDT_2$	1.579	−1.129	0.0772	1.996	0.2756	0.5101	0.4743
		$HSDT_4$	1.441	−1.100	0.0718	1.846	0.2573	0.5525	0.4400
5	5	Elast.	2.205	−1.040	—	1.763	0.2626	0.4260	0.4016
		$HSDT_1$	2.214	−1.051	1.037	1.760	0.2596	0.4257	0.3784
		$HSDT_2$	2.187	−1.036	1.039	1.735	0.2565	0.4022	0.3639
		$HSDT_4$	1.896	−0.964	0.967	1.547	0.2293	0.3922	0.3511
	10	Elast.	1.020	−0.8340	—	1.047	0.1660	0.4160	0.2425
		$HSDT_1$	1.021	−0.8315	0.8262	1.044	0.1652	0.4332	0.2332
		$HSDT_2$	1.008	−0.8223	0.8253	1.030	0.1634	0.4306	0.2307
		$HSDT_4$	0.931	−0.8037	0.8064	0.962	0.1543	0.4068	0.2235

Table 3. Nondimensional deflections of laminated spherical deep and shallow shells

R/a	a/h	Theory	$N = 3$	$N = 4$	$N = 5$
1	5	$HSDT_1$	1.208	1.179	1.151
		$HSDT_2$	1.079	1.054	1.025
	10	$HSDT_1$	0.3761	0.3748	0.3615
		$HSDT_2$	0.3475	0.3467	0.3328
2	5	Elast.	1.482	1.434	1.376
		$HSDT_1$	1.482	1.433	1.379
		$HSDT_2$	1.422	1.376	1.324
		$HSDT_3$	1.420	1.228	1.217
	10	Elast.	0.6087	0.6128	0.5671
		$HSDT_1$	0.6090	0.6085	0.5670
		$HSDT_2$	0.5877	0.5875	0.5468
		$HSDT_3$	0.5840	0.5673	0.5344
5	5	Elast.	1.549	1.495	1.417
		$HSDT_1$	1.546	1.488	1.425
		$HSDT_2$	1.534	1.478	1.414
		$HSDT_4$	1.461	1.240	1.228
	10	Elast.	0.7325	0.7408	0.6707
		$HSDT_1$	0.7340	0.7345	0.6708
		$HSDT_2$	0.7287	0.7293	0.6660
		$HSDT_4$	0.6905	0.6664	0.6182

Table 4. Comparison between the present shell theory ($HSDT_1$) and the shallow shell theory ($HSDT_2$) ($N = 3, S = 5$)

R_1/a		1.0	1.5	2.0	2.5	3.0	5.0	Plate
				cylindrical shells ($b/a = 3$)				
$\bar{w}$	$HSDT_1$	2.699	2.354	2.235	2.177	2.144	2.089	2.033
	$HSDT_2$	2.194	2.150	2.125	2.108	2.096	2.072	2.033
$\bar{\sigma}_{1-}$	$HSDT_1$	−1.241	−1.142	−1.102	−1.082	−1.069	−1.046	−1.018
	$HSDT_2$	−1.093	−1.073	−1.061	−1.053	−1.048	−1.036	−1.018
				spherical shells ($b/a = 1$)				
$\bar{w}$	$HSDT_1$	1.208	1.404	1.482	1.516	1.532	1.546	1.516
	$HSDT_2$	1.079	1.316	1.422	1.474	1.502	1.534	1.516
$\bar{\sigma}_{1-}$	$HSDT_1$	−0.4699	−0.6098	−0.6706	−0.7006	−0.7171	−0.7399	−0.7447
	$HSDT_2$	−0.4575	−0.5873	−0.6509	−0.6846	−0.7041	−0.7330	−0.7447

can be obtained only by integrating the three-dimensional equilibrium equations, otherwise σ_4 and σ_5 calculated from the constitutive equations are not continued at layer interfaces.

6. CONCLUSIONS

A refined theory for laminated shells with higher-order transverse shear deformation is developed. The in-plane displacements are improved by imposing a series of conditions and the number of displacement unknowns are reduced to five. The influence of the materials and plyup patterns on the displacement field is included. The present theory ensures the continuity of interlaminar transverse shear stresses and the compatibility conditions on the external bounding surfaces. The equilibrium equations and boundary conditions are proposed. The closed-form solutions for simply-supported cross-ply shells are presented. The numerical examples show that present theory can predict very accurate results for both deep and shallow shells. Moreover, as the number of the displacement unknowns is only five, the present theory is relatively simple and can be conveniently applied to numerical analysis methods (e.g., FEM).

REFERENCES

Dennis, S. T. and Palazotto, A. N. (1991). Laminated shell in cylindrical bending, two-dimensional approach vs exact. *AIAA J.* **29,** 647–650.

Di Sciuva, M. (1987). An improved shear-deformation theory for moderately thick multilayered anisotropic shells and plates. *J. Appl. Mech.* **54,** 589–596.

Dong, S. B. and Tso, F. K. W. (1972). On a laminated orthotropic shell theory including transverse shear deformation. *J. Appl. Mech.* **39,** 1091–1096.

He, L. H. (1994). A linear theory of laminated shells accounting for continuity of displacements and transverse shear stresses at layer interfaces. *Int. J. Solids Struct.* **31,** 613–627.

Huang, N. N. (1994). Influence of shear correction factors in the higher-order shear deformation laminated shell theory. *Int. J. Solids Struct.* **31,** 1263–1227.

Jing, H. S. and Tzeng, K. G. (1993). Refined shear deformation theory of laminated shells. *AIAA J.* **31,** 765–773.

Librescu, L., Khdeir, A. A. and Frederick, D. (1989). A shear deformation theory of laminated composite shallow shell-type panel and their response analysis I: free vibration and buckling. *Acta. Mech.* **76,** 1–33.

Noor, A. K. and Peters, J. M. (1990). A posteriori estimates for shear correction factors in multi-layered composite cylinders. *J. Engng Mech.* **115,** 1225–1244.

Reddy, J. N. and Liu, C. F. (1985). A higher-order shear deformation theory of laminated elastic shells. *Int. J. Engng Sci.* **23,** 319–330.

Shu, X. P. and Sun, L. X. (1994a). An improved simple higher-order theory for laminated composite plates. *Computers and Struct.* **50,** 231–236.

Shu, X. P. and Sun, L. X. (1994b). Thermomechanical buckling of laminated composite plates with higher-order transverse shear deformation. *Computers and Struct.* **53,** 1–7.

Shu, X. P. (1996). An improved simple higher-order theory for laminated composite shells. *Computers and Struct.* (to be published).

Soldatos, K. P. and Timarci, T. (1993). Unified formulation of laminated composite, shear deformation, five-degrees-of-freedom cylindrical shell theory. *Comp. Struct.* **25,** 165–171.

Touratier, M. (1992). A refined theory of laminated shallow shells. *Int. J. Solids Struct.* **29,** 1401–1415.

Whitney, J. M. and Sun, C. T. (1947). A refined theory for laminated anisotropic cylindrical shells. *J. Appl. Mech.* **41,** 471–476.

APPENDIX

Coefficients of matrix $[K_{ij}]$ are defined as follows:

$$K_{11} = (A_{11}+2B_{11}/R_1+D_{11}/R_1^2)\alpha^2+(A_{33}+2B_{32}/R_1+D_{22}/R_1^2)\beta^2+C_{11}/R_1^2$$

$$K_{12} = (A_{12}+A_{33}+B_{21}/R_1+B_{32}/R_1+B_{14}/R_2+B_{33}/R_2+D_{14}/R_1R_2+D_{23}/R_1R_2)\alpha\beta$$

$$K_{13} = -(B_{11}+D_{11}/R_1)\alpha^3-(B_{14}+B_{32}+B_{33}+D_{14}/R_1+D_{22}/R_1+D_{23}/R_1)\alpha\beta^2$$

$$-(A_{11}/R_1+A_{12}/R_2+B_{11}/R_1^2+B_{21}/R_1R_2+C_{11}/R_1)\alpha$$

$$K_{14} = (F_{11}+E_{11}/R_1)\alpha^2+(F_{32}+E_{22}/R_1)\beta^2-C_{11}/R_1$$

$$K_{15} = (F_{14}+F_{33}+E_{14}/R_1+E_{23}/R_1)\alpha\beta$$

$$K_{22} = (A_{33}+2B_{33}/R_2+D_{33}/R_2^2)\alpha^2+(A_{22}+2B_{24}/R_2+D_{44}/R_2^2)\beta^2+C_{22}/R_2^2$$

$$K_{23} = -(B_{24}+D_{44}/R_2)\beta^3-(B_{21}+B_{32}+B_{33}+D_{14}/R_2+D_{23}/R_2+D_{33}/R_2)\alpha^2\beta$$

$$-(A_{12}/R_1+A_{22}/R_2+B_{14}/R_1R_2+B_{24}/R_2^2+C_{22}/R_2)\beta$$

$$K_{24} = (F_{21}+F_{32}+E_{32}/R_2+E_{41}/R_2)\alpha\beta$$

$$K_{25} = (F_{33}+E_{33}/R_2)\alpha^2+(F_{24}+E_{44}/R_2)\beta^2-C_{22}/R_2$$

$$K_{33} = D_{11}\alpha^4+(D_{22}+D_{33}+2D_{14}+2D_{23})\alpha^2\beta^2+D_{44}\beta^4+(2B_{11}/R_1+2B_{21}/R_2+C_{11})\alpha^2$$

$$+(2B_{14}/R_1+2B_{24}/R_2+C_{22})\beta^2+A_{11}/R_1^2+2A_{12}/R_1R_2+A_{22}/R_2^2$$

$$K_{34} = -E_{11}\alpha^3-(E_{22}+E_{32}+E_{41})\alpha\beta^2-(F_{11}/R_1+F_{21}/R_2-C_{11})\alpha$$

$$K_{35} = -(E_{14}+E_{23}+E_{33})\alpha^2\beta-E_{44}\beta^3-(F_{14}/R_1+F_{24}/R_2-C_{22})\beta$$

$$K_{44} = G_{11}\alpha^2+G_{22}\beta^2+C_{11}$$

$$K_{45} = (G_{14}+G_{23})\alpha\beta$$

$$K_{55} = G_{33}\alpha^2+G_{44}\beta^2+C_{22}.$$